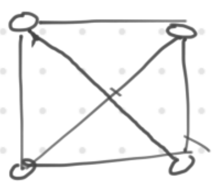
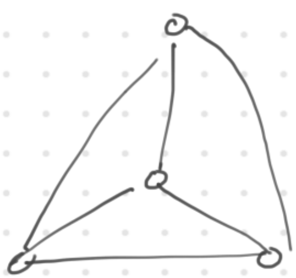
Planarity - algorithmically testing planarity.

Def a graph is planar if it can be drawn in the plane without crossing edges

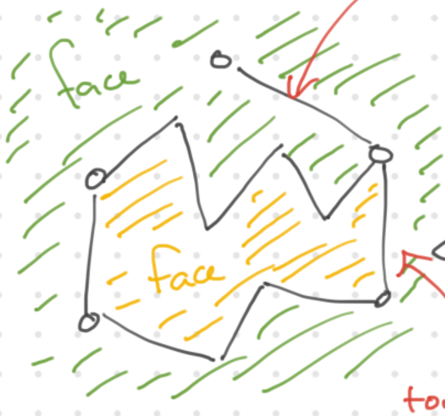
$K_4 =$  ← has crossing edges

but $\exists$ a drawing without crossing edges



To provide a certificate that a graph is planar, it suffices to explicitly give an embedding but ~~how~~ what certificates do we have to show a graph is not planar.

Def a plane graph G is a graph drawn in the plane without crossing edges. A face of the drawing is a region of $\mathbb{R}^2 - G$



G drawn in the plane

we assume edges are drawn as union of straight line segments

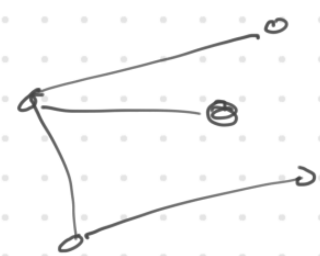
<u>Prop</u> every edge touches at most two faces, + every edge in a cycle touches exactly 2 faces

<u>Thm</u> (Euler) G a connected planar graph w/ n vertices, e edges + f faces, Then

$$n - e + f = 2$$

<u>pf</u> by induction on e

<u>Base case</u> $n-1 = e \Rightarrow G$ is a tree (cause it's connected w/ $n-1$ edges) $\Rightarrow$ ~~$n-1$ vertices,~~ n vertices, $n-1$ edges + how many faces?



exactly one face (and this is true for all trees)

$n - (n-1) + 1 = 2$ as desired. ✓

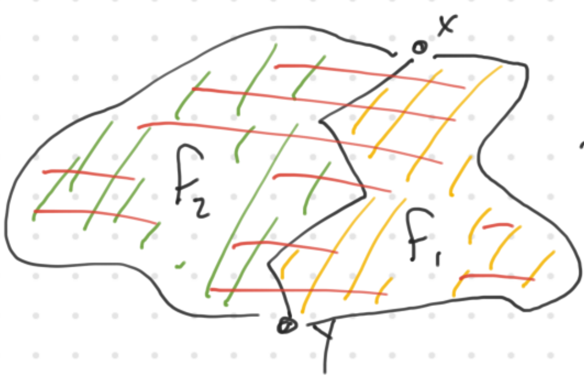
in general, assume G has e edges for $e \geq n$.

$\Rightarrow$ G contains a cycle C w/ an edge xy.

$G - xy$ is still a plane graph. + it is connected because deleting an edge in a cycle leaves a conn. graph.

So we can apply induction to $G - xy$.

How many faces does $G - xy$ have?



The edge xy touches two distinct faces: f_1 + f_2

deleting xy has the effect of merging f_1 + f_2 into a single face

$\Rightarrow$ $G - xy$ has $f - 1$ faces

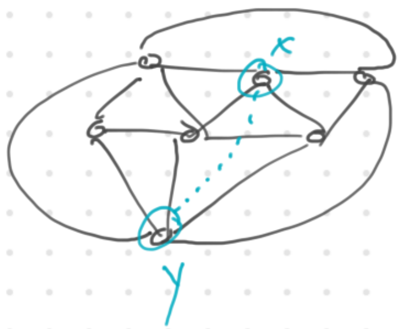
By induction, we see that $|V(G-xy)| - |E(G-xy)| + \#\text{faces}(G-xy) = 2$

$\Rightarrow$ $n - (e-1) + (f-1) = 2$

$\Rightarrow$ $n - e + f = 2$, as desired. $\blacksquare$

Let G have $n \geq 3$ vertices

& assume G is <u>maximally</u> <u>planar</u>
ie for any pair of vertices $x, y \in V(G)$ st $x \neq y$, we have that $G + xy$ is not planar.

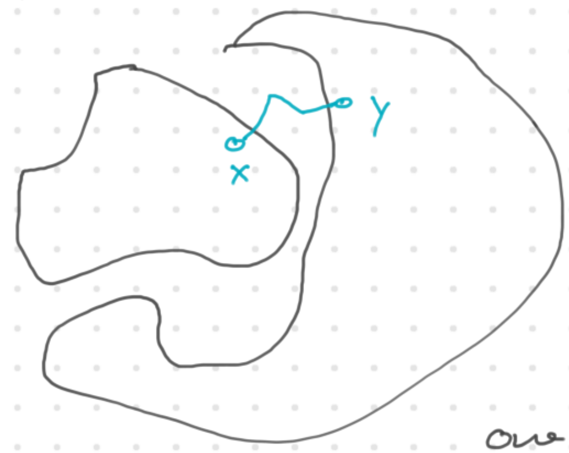


maximally planar graph

we want an edge bound on # of edges in a maximally planar graph.

<u>OB</u> If G is connected and maximally planar w/ ≥ 3 vertices. Then G does not have a bridge.

<u>pf</u> G is connected

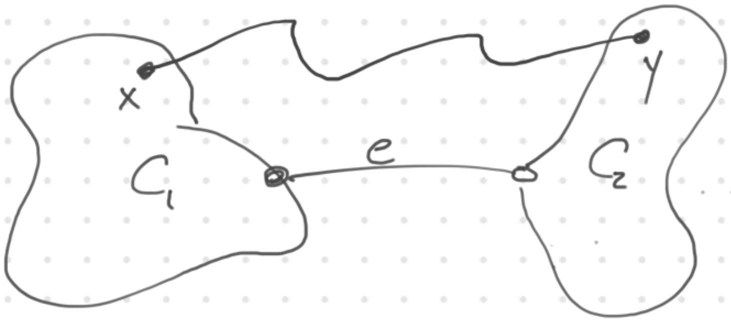


if we had more than one component pick x touching infinite face on one & y touching infinite face on the other & add edge xy keeping $G + xy$ planar

$\Rightarrow\Leftarrow$ to maximally planar.

So we conclude G is conn.

<u>If</u> G has a bridge e



look at two components of $G-e$, say C_1 + C_2, + pick x touching infinite face of C_1 + y touching inf face of C_2 + we can add edge xy keeping the graph planar $\rightarrow\leftarrow$

Conclude: G is maximally planar, w/ at least 3 vertices

Then every face touches ≥ 3 edges + every edge sees two faces.

plug this into Euler's formula

$$n - e + f = 2$$

$$3 \cdot f \leq \sum_{h \text{ a face}} \underbrace{\#\text{ of edges touching } h}_{\geq 3 \text{ for each face } h} = 2e$$

$$\Rightarrow \quad f \leq \tfrac{2}{3} e$$

subbing in to the formula

$$2 = n - e + f \leq n - e + \tfrac{2}{3} e$$

$$\tfrac{1}{3} e \leq n - 2 \quad \Rightarrow \quad e \leq 3n - 6$$

Conclusion: if G is a maximally planar graph w/ ≥ 3 vertices, then $|E(G)| \leq 3n-6$

$\Rightarrow$ every planar graph on n vertices with $n \geq 3$ has at most $3n-6$ edges.

(because every planar graph is a subgraph of a maximally planar graph w/ the same # of vertices)

This gives a certificate that a graph is not planar.

Cor K_5 is not planar.

pf $n=5$ $e = \binom{5}{2} = 10$

$3 \cdot 5 - 6 = 9 < 10$

$\Rightarrow K_5$ is not planar.

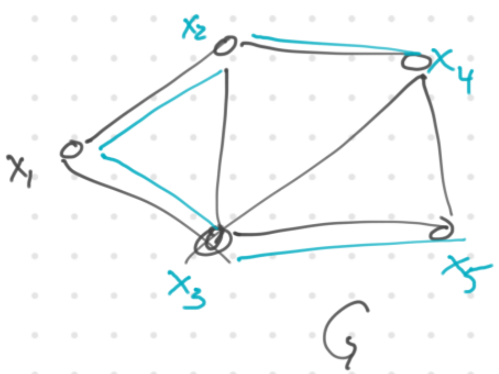
Algorithmically, in a planarity testing algorithm, we can do a pass + count # of edges + if $e \geq 3n-5$, return "NOT PLANAR"

st-labelings

Prop G connected. $\exists$ an ordering $x_1, x_2, \ldots, x_n$ of $V(G)$ st $\forall\ i \geq 2$

The vertex x_i has a nbr in $x_1, x_2, \ldots, x_{i-1}$

pf ~~perform~~ Fix x_1, start a BFS/DFS, etc, an as you discover a new vertex, add it to the end of the ordering



in this ordering, every vertex is after it's father in the seach tree $\Rightarrow$ every vertex has a nbr to the left.

Prop If G is not connected, Then $\nexists$ an ordering $x_1, x_2, \ldots, x_n$ of $V(G)$ st $\forall\, i \geq 2$ ~~$\exists$~~ x_i has a nbr in $\{x_1, \ldots, x_{i-1}\}$

pf pick any ordering of $V(G)$ $x_1, x_2, \ldots, x_n$ & pick a component which does not contain x_1. Call it C & let

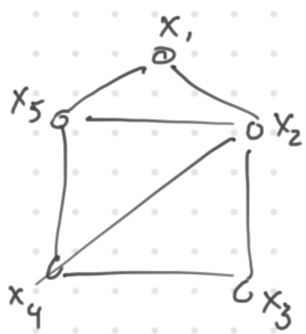
$$i := \min_i x_i \in V(C)$$

Then x_i does not have a nbr in $\{x_1, \ldots, x_{i-1}\}$.

Def an s-t ordering of

a graph G is an ordering $x_1, x_2, \dots, x_n$ of $V(G)$ st

- $x_1 \sim x_n$
- $\forall i$, $2 \le i \le n-1$, x_i has a nbhr in $\{x_1, \dots, x_{i-1}\}$ **AND** in $\{x_{i+1}, \dots, x_n\}$

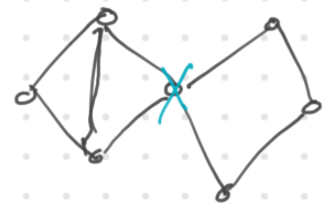


s-t labeling of the graph.

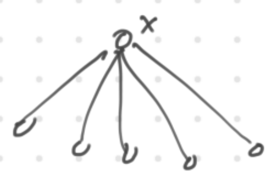
Prop if G is not $\underbrace{\text{2-conn}}$ then $\nexists$ an s-t ordering.

↳ $|V(G)| \ge 3$ & $\forall x \in V(G)$ $G - x$ is conn.

pf



If G is not 2-conn, then $\exists$ a vertex x st $G - x$ has ≥ 2 comps C_1 & C_2



pick an ordering $x_1, \dots, x_n$ st $x_1 \sim x_n$

Since $x_1 \sim x_n$, either C_1 or C_2 is disjoint from $\{x_1, x_n\}$

Say C_1

vertices in C_1

x_i

$x_1 \quad x_2 \quad x_3 \quad \cdots \quad \circ \quad x_n$

$x_{i'}$

let i smallest index st $x_i \in V(C_1)$

i' largest index st $x_{i'} \in V(C_1)$

if x_i has a nbr in $x_1, \dots, x_{i-1}$, That neighbor must be x $\Rightarrow x \in \{x_1, \dots, x_{i-1}\}$

if $x_{i'}$ has a nbr in $x_{i'+1}, \dots, x_n$, Then $x \in \{x_{i'+1}, \dots, x_n\}$

These can't both be true That $x \in \{x_1, \dots, x_{i-1}\}$ & $x \in \{x_{i'+1}, \dots, x_n\}$

$\Rightarrow x_1, \dots, x_n$ is NOT an st ordering.

Prop if G is 2-conn, Then $\exists$ an s-t ordering

Problem: Given a network G w/ disjoint sets of vertices $S=\{s_1,\dots,s_k\}$, $T=\{t_1,\dots,t_\ell\}$, define an S-T flow $f: E(G)\to\mathbb{R}$

ST

- $f(u,v) = -f(v,u)$
- $f(u,v) \le c(u,v)$ $\forall$ edges
- $\forall\, x \notin S\cup T$,
$$\sum_{\substack{(x,y):\\(x,y)\in E(G)}} f(x,y) = 0$$

value of flow is $\displaystyle\sum_{\substack{(u,v)\\ u\in S,\ v\notin S}} f(u,v)$

give an alg to find max flow

Problem ~~given~~ given network G, $s,t\in V(G)$, + $c: E(G)\to\mathbb{R}$ find a max flow f from s to t + an s-t cut of same capacity with minimum # of edges over all such cuts